Dispersive model equations for fluid flow

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One-dimensional Wave Models

Solitary waves occur in wave propagation systems having a weak nonlinearity and dispersion, such as Korteweg and de Vries :

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1}$$

and Peregrine & Benjamin, Bona & Mahoney equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{2}$$

obtained from KdV by using $u_t \approx -u_x$.

Mathematical Structure

KdV is a completely integrable Hamiltonian system (whereas PBBM is not). Solitary wave interactions are particle-like for KdV (*solitons*), not for PBBM . However, Camassa and Holm equation does have the soliton property:

$$u_t + u_x + uu_x - \alpha u_{xxt} = \frac{2}{3}\alpha u_x u_{xx} + \frac{1}{3}\alpha u u_{xxx} \qquad (3)$$

Here, α is a positive constant indicating the amount of dispersion; previously we just had $\alpha = 1$.

From modeling perspective, terms on right-hand side of (3) are even smaller than nonlinear and (linear) dispersive terms on left-hand side.

The CH equation (3) has a suggestive form

$$\begin{aligned} v_t + u_x + u v_x + 2 v u_x &= 0 \\ u &= \Big(1 - \alpha \frac{\partial^2}{\partial x^2}\Big)^{-1} v \end{aligned} \tag{4}$$

Means u solves $\Big(1 - \alpha \frac{\partial^2}{\partial x^2}\Big) u = v$ with $u \to 0$ at ∞.

CH equation (3/4) is a nonlinear perturbation of PBBM and so also of KdV and can be viewed as a nonlinear advection equation for a variable v which has a simple relation to the advection velocity u.

(4) forms the template for a three-dimensional fluid flow model including dispersion.

3-D dispersive fluid models

The 3-D analog of the Camassa-Holm equation is

$$\mathbf{v}_t - \nu\Delta\mathbf{u} + (\mathbf{curl}\,\mathbf{v}) \times \mathbf{u} + \nabla p = 0 \qquad (5)$$

where $(1-\alpha\Delta)\,\mathbf{u} = \mathbf{v}$, $\alpha > 0$ and $\nabla \cdot \mathbf{u} = 0$.

$\alpha = 0$ implies $\mathbf{u} = \mathbf{v}$; (5) becomes Navier-Stokes

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{curl}\,\mathbf{u}) \times \mathbf{u} + \nabla p = 0 \qquad (6)$$

where pressure $\tilde{p} = p - \frac{1}{2}\mathbf{u}\cdot\mathbf{u}$.

Nonlinear term $(\mathbf{curl}\,\mathbf{u}) \times \mathbf{u}$ differs from usual one $\mathbf{u}\cdot\nabla\mathbf{u}$ via this transformation involving a re-definition of the "pressure" variable.

α model history

The α-model equation (5) first appeared as a model of Rivlin and Ericksen: a continuum of material with velocity $\mathbf{u}$ described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \nabla \cdot \mathbf{T}, \tag{7}$$

where $\mathbf{T} = \mathbf{T}(\mathbf{u})$ is the stress and we define

$$\frac{\mathrm{d}}{\mathrm{d}t}w := w_t + \mathbf{u} \cdot \nabla w \tag{8}$$

for any w (either scalar, vector or tensor valued).

Rivlin-Ericksen grade n model

Assume that the stress tensor $\mathbf{T}$ has the form (for the grade n model)

$$\mathbf{T} = -\tilde{p}\mathbf{I} + \mathcal{S}^n(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \tag{9}$$

where $\tilde{p} =$ pressure and $\mathbf{a}_j$ are Rivlin-Ericksen tensors defined recursively by (recall the notation (8))

$$\mathbf{a}_1 = \mathbf{L} + \mathbf{L}^T \quad , \quad \mathbf{L} = \nabla\, \mathbf{u}\,,$$

$$\mathbf{a}_j = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}_{j-1} + \mathbf{a}_{j-1}\mathbf{L} + \mathbf{L}^T\mathbf{a}_{j-1}\,,$$

$\mathcal{S}^n = \sum_{i=1}^n \mathbf{S}_i$ and each $\mathbf{S}_i$ is a polynomial in the $\mathbf{a}_j$:

$$\mathbf{S}_1 = \eta\mathbf{a}_1 \ , \ \mathbf{S}_2 = \alpha_1\mathbf{a}_2 + \alpha_2\mathbf{a}_1^2\,, \text{ etc}\,.$$

Grade 2 Model

$$\mathbf{a}_1 = \nabla \mathbf{u} + \nabla \mathbf{u}^T \,, \quad \mathbf{a}_2 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}_1 + \mathbf{a}_1\mathbf{L} + \mathbf{L}^T\mathbf{a}_1 \,,$$

$$\mathbf{T} = \mathbf{T}^2 = -\tilde{p}\mathbf{I} + \eta\mathbf{a}_1 + \alpha_1\mathbf{a}_2 + \alpha_2\mathbf{a}_1^2 \,, \tag{10}$$

where the parameters η, α_i are material constants. Physical arguments show that

$$\eta \geq 0 \quad , \quad \alpha_1 \geq 0 \quad \text{and } \alpha_1 + \alpha_2 = 0 \,.$$

Setting $\alpha = \alpha_1$ (and $\alpha_2 = -\alpha$) and substituting (10) into (7) yields the dispersive fluid equations (5).

Geometry of maps

When $\nu = 0$ in (5), we get a grade-two variant of the Euler equations. Consider the class $\mathcal{V}^s$ of vector fields $\mathbf{u}$ satisfying (a) $\mathbf{u} \in H^s(\Omega)$ (b) $\nabla \cdot \mathbf{u} = 0$ and (c) $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Given $\mathbf{u}(\cdot, t) \in \mathcal{V}^s$, consider the flow $\mathbf{f_u} = \mathbf{f_u}(\mathbf{x}, t)$ generated by $\mathbf{u}$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f_u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{f_u}(\mathbf{x}, t), t) \qquad (11)$$

where $\mathbf{f_u}(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in \Omega$. For each t, $\mathbf{f_u}(\cdot, t) \in \mathcal{D}_\Omega =$ the space of volume preserving diffeomorphisms of Ω. Thus the map $t \to \mathbf{f_u}(\cdot, t)$ is a curve in $\mathcal{D}_\Omega$.

Geometry of $\mathcal{D}_\Omega$

$\mathcal{D}_\Omega$ has a natural group structure: composition. The tangent space $T_{\mathcal{D}_\Omega}(\mathcal{I})$ to $\mathcal{D}_\Omega$ at $\mathcal{I}$ can be identified with the space $\mathcal{V}^s$ of divergence-free vector fields. Putting an inner-product on $\mathcal{V}^s$ puts a metric on the tangent space to $\mathcal{D}_\Omega$ at $\mathcal{I}$. For example,

$$< \tau, \sigma >_{L^2} = \int_\Omega \tau(\mathbf{x}) \cdot \sigma(\mathbf{x}) \, d\mathbf{x} \tag{12}$$

$$< \tau, \sigma >_{H^1_\alpha} = \int_\Omega \tau(\mathbf{x}) \cdot \sigma(\mathbf{x}) + \alpha \nabla \tau(\mathbf{x}) : \nabla \sigma(\mathbf{x}) \, d\mathbf{x}. \tag{13}$$

Using the group structure allows us to translate this metric to the entire tangent bundle invariantly.

Geodesics come from solutions

Remarkably $\mathbf{u}$ solves the Euler equations if and only if the curve $t \to \mathbf{f_u}(\cdot, t)$ is a geodesic in $\mathcal{D}_\Omega$ with metric (12) given by the L^2 inner-product on $\mathcal{V}^s$.

Even more remarkably, $\mathbf{u}$ solves α-model (5) if and only if the curve $t \to \mathbf{f_u}(\cdot, t)$ is a geodesic in $\mathcal{D}_\Omega$ with metric (13) given by the H^1_α inner-product on $\mathcal{V}^s$.

This structural property of the α-model (5) exhibits a key property of a good model and makes it clear that it does not appear by chance.

Turbulence models

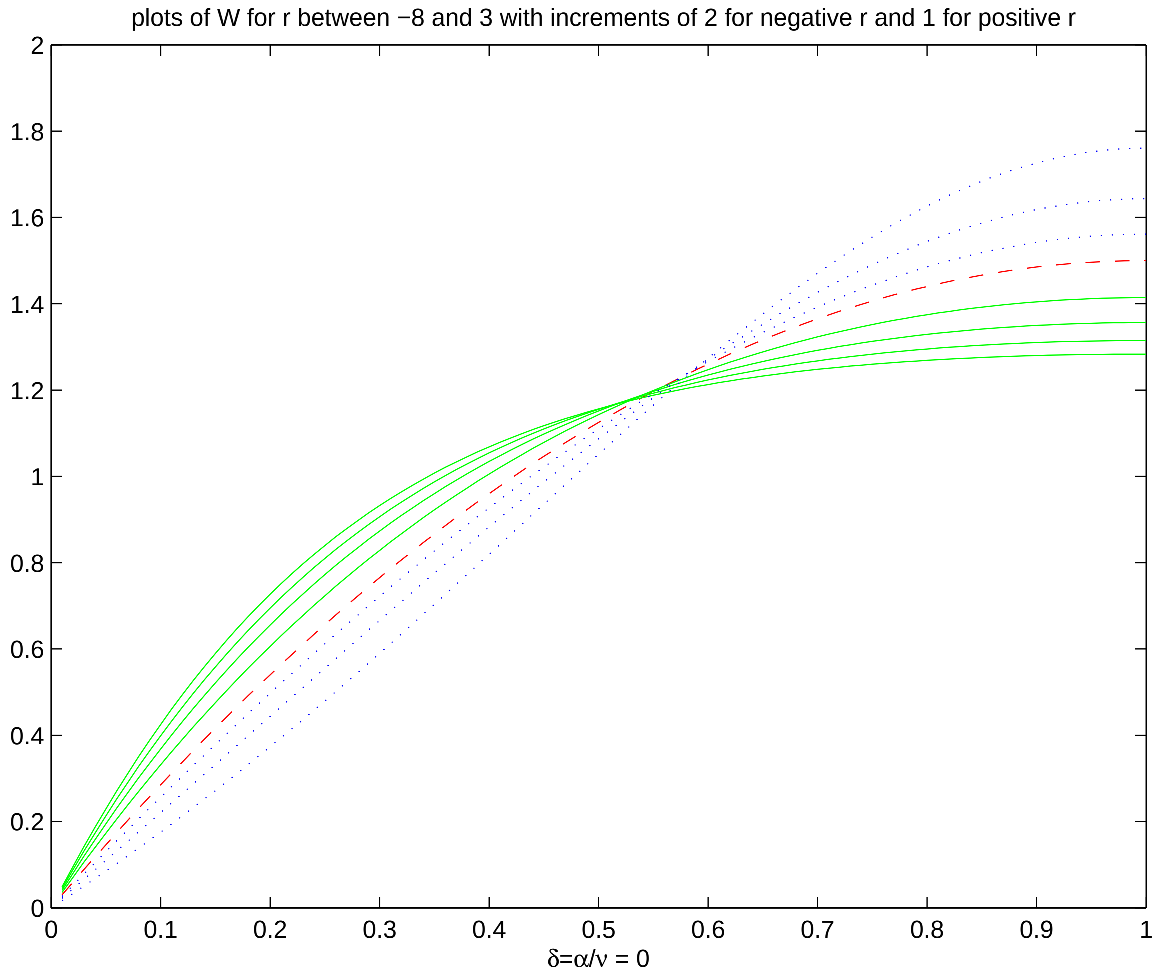
It is of interest to study statistical properties of ensembles of solutions of the Navier-Stokes equations (meaning (6) together with $\nabla \cdot \mathbf{u} = 0$). If we let $\mathbf{u}$ denote such an ensemble average of solutions, then at least one such averaged equation takes the form (7). Given the general considerations leading to (9), it is not surprising that appropriate models would lead ultimately to (5). A slightly modified model has been shown in to provide an accurate model of turbulence experiments done in a channel.

Lubrication model solutions

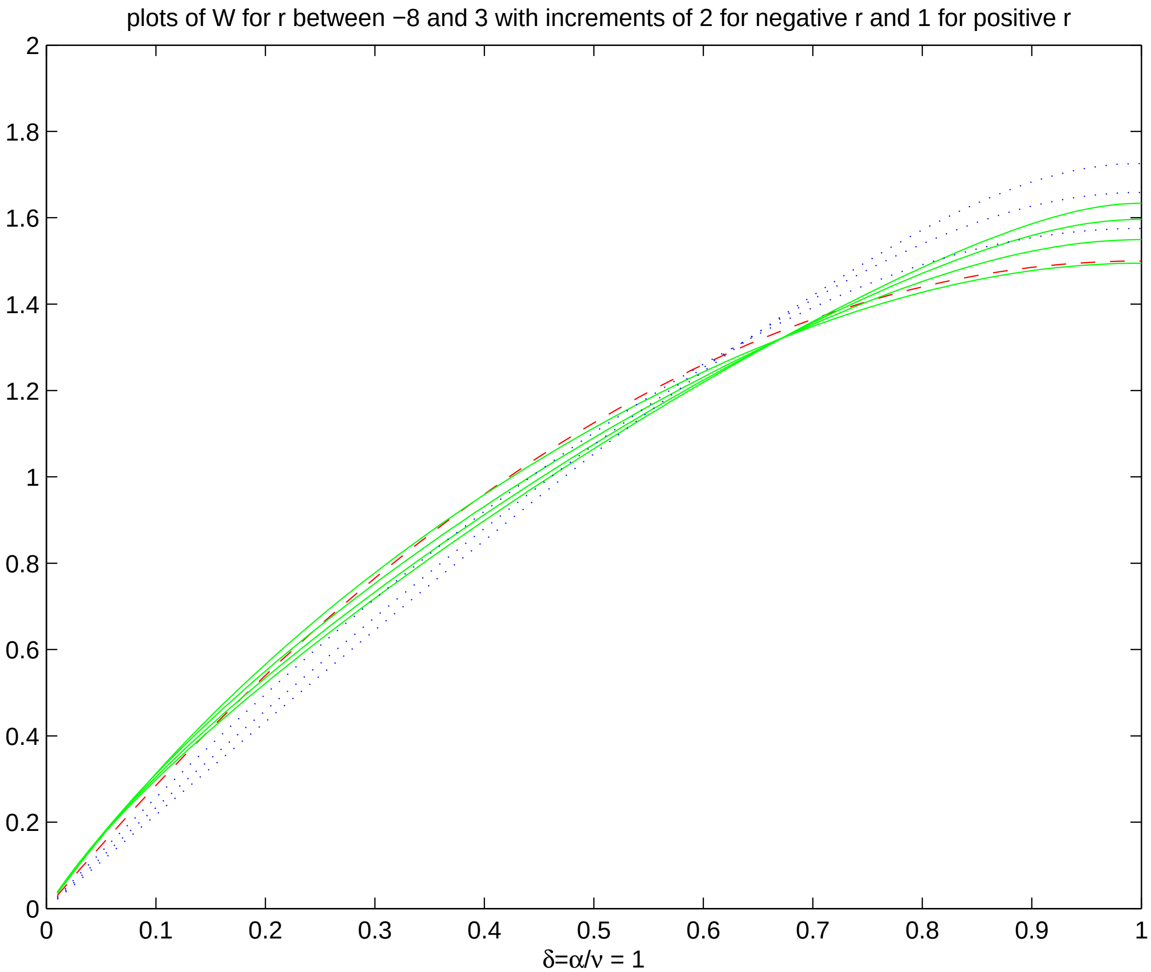
Flow in a long channel whose horizontal velocity has the form shown in the figures.

The shape depends on the parameter $r := t'R$ where R is the Reynolds number and t is the thickness of the channel.

When $\alpha > 0$ the effect of R is diminished.



plots of W for r between −8 and 3 with increments of 2 for negative r and 1 for positive r
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
δ=α/ν = 0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2
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0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
δ=α/ν = 1

Stability of the grade-two model

Taking the $\mathbf{curl}$ of (5) and introducing the variable

$$\mathbf{z} = \mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,(\mathbf{u} - \alpha\Delta\mathbf{u}) \qquad (14)$$

This leads to a transport equation for $\mathbf{z}$:

$$\alpha\mathbf{z}_t + \nu\mathbf{z} + \alpha\mathbf{u}\cdot\nabla\mathbf{z} - \alpha\mathbf{z}\cdot\nabla\mathbf{u} = \nu\,\mathbf{curl}\,\mathbf{u} \qquad (15)$$

The steady versions of (5) and (15) in 2-D read

$$\begin{aligned} -\nu\Delta\mathbf{u} + z(u_2 - u_1) + \nabla p &= 0 \\ \nabla\cdot\,\mathbf{u} &= 0 \qquad (16) \\ \nu z + \alpha\mathbf{u}\cdot\nabla z &= \nu\,\mathrm{curl}\,\mathbf{u} \end{aligned}$$

where we now $\mathbf{u}$ denotes a 2-vector valued function and $\mathrm{curl}\,\mathbf{u} := u_{1,2} - u_{2,1}$.

Transport equation

Let us write the general form of the transport equation in (16), after dividing by ν, as

$$z + \mathcal{W}\mathbf{u} \cdot \nabla z = f. \tag{17}$$

If all we know is that $\mathbf{u} \in H^1$, then $f \in L^2$ is the best we can hope for in (16). But then we could not hope for more than $z \in L^2$ either, as (17) provides no smoothing. And for $z \in L^2$ (and $\mathbf{u} \in H^1$), the term $\mathbf{u} \cdot \nabla z$ is a concern.

Certainly, $\mathbf{u} \cdot \nabla z$ will not in general be in L^2 for unrelated z and $\mathbf{u}$.

Transport equation solution

Miraculously, it is possible to show that (provided $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$), the problem (16), i.e.,

$$z + \mathcal{W}\mathbf{u} \cdot \nabla z = f$$

has a unique solution $z \in L^2$ for any $f \in L^2$.

In fact, you can say more, in that z lies in the space

$$X_{\mathbf{u}} := \left\{ w \in L^2 \ : \ \mathbf{u} \cdot \nabla w \in L^2 \right\} \tag{18}$$

and $\|z\|_{L^2} \leq \|f\|_{L^2}$.

Conclusions

We have examined a model for dispersive flow in two and three dimensions. We described several ways in which this model can be derived.

We have shown that stability can be established in the 2-D case. Numerical schemes for approximating the equations and their corresponding stability and convergence properties have been proved.

Lubrication models indicate behavior of laminar flows.